Exam Calculus 1

October 30, 2015: 8.30-11.30.

This exam has 8 problems. Each problem is worth 1 point. Total: 8 + 1 (free) = 9 points; more details can be found below. Write on each page your name and student number. The use of annotations, books and calculators is not permitted in this examination. All answers must be supported by work. Success.

1. (a) Formulate the principle of mathematical induction.

   (b) If functions f and g are differentiable at x, then the function $f+g$ is differentiable at x and $(f+g)'(x) = f'(x) + g'(x)$. Prove, using the principle of mathematical induction, that the rule for differentiation sums extends to sums of any finite number of terms:

   $$(f_1 + f_2 + \cdots + f_n)' = f_1' + f_2' + \cdots f_n'$$

   where $n \geq 1$ denotes an arbitrary integer and the functions f_i are differentiable at x for $i = 1, 2, ..., n$.

2. Find all (complex) solutions z of

   $$\mathrm{e}^{-iz} \;=\; \frac{i+1}{i-1}$$

   and plot them in the complex plane.

3. (a) The function f is defined on some open interval that contains the number a, except possibly at a itself. Give the precise definition of

   $$\lim_{x \to a} f(x) \;=\; L$$

   (b) Prove, using this definition, that

   $$\lim_{x \to 3} x^2 \;=\; 9$$

4. (a) Formulate the Mean Value Theorem.

   (b) Use the Mean Value Theorem to prove the inequality

   $$|\sin a - \sin b\,| \;\leq\; |a - b|$$

   for all a and b.

5. The function f is given by

$$f(x) = \mathrm{e}^{-1/x^2}$$

if $x \neq 0$ and $f(0) = 0$.

(a) Use the definition of derivative to compute $f'(0)$.

(b) Is $f'(x)$ differentiable at $x = 0$?

6. (a) Evaluate

$$\lim_{x \to 0} (1 - 2x)^{1/x}$$

(b) The function f is differentiable on $(-\infty, \infty)$. Determine $f(0)$ in the following cases:

$$(i) \int_x^0 f(t)dt = \mathrm{e}^{x^2} \qquad (ii) \int_0^{x^2} f(t)dt = \mathrm{e}^{x^2}$$

7. Evaluate

(a)

$$\int \sqrt{1+x^2}\ x^5\ dx$$

(b)

$$\int_1^e (\ln x)^2\ dx$$

8. Solve the differential equation

$$xy' - 2y = x^2$$

Maximum points:

1a	0.4	2	1.0	3a	0.4	4a	0.4	5a	0.5	6a	0.5	7a	0.5	8	1.0
b	0.6			b	0.6	b	0.6	b	0.5	b	0.5	b	0.5		